

Vortices and invariant surfaces generated by symmetries for the 3D Navier–Stokes equations

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Abstract

We show that certain infinitesimal operators of the Lie–point symmetries of the incompressible 3D Navier–Stokes equations give rise to vortex solutions with different characteristics. This approach allows an algebraic classification of vortices and throws light on the alignment mechanism between the vorticity $\vec{\omega}$ and the vortex stretching vector $S \vec{\omega}$, where S is the strain matrix. The symmetry algebra associated with the Navier–Stokes equations turns out to be infinite–dimensional. New vortical structures, generalizing in some cases well–known configurations such as, for example, the Burgers and Lundgren solutions, are obtained and discussed in relation to the value of the dynamic angle $\phi = \arctan \frac{[\vec{\omega} \wedge S \vec{\omega}]}{\vec{\omega} \cdot \vec{\omega}}$. A systematic treatment of the boundary conditions invariant under the symmetry group of the equations under study is also performed, and the corresponding invariant surfaces are recognized.

1 Introduction

Let us consider the incompressible 3D Navier–Stokes equations

$$\Delta_1 \equiv \vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \nabla p - \nu \nabla^2 \vec{u} - \vec{f} = 0, \quad (1.1)$$

$$\Delta_2 \equiv \nabla \cdot \vec{u} = 0, \quad (1.2)$$

where $\vec{u} = (u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t))$ is the velocity field, $p = p(x, y, z, t)$ the fluid pressure, ν the viscosity coefficient, and $\vec{f} = \vec{f}(x, y, z, t)$ is an external force.

Generally, in the studies of turbulence flows it is important to recognize the generation mechanisms and evolution of vortical structures. In this task both analytical and numerical methods are of help. For example, Tanaka and Kida [1] showed that high vorticity with relatively low strain rate corresponds to the vortex tube, while high vorticity with comparable strain rate to the vortex sheet. Wray and Hunt [2] presented an algorithm for classification of turbulent structures where the flow field is classified into four regions (eddies, convergence, shear and stream regions).

Interesting results for the 3D Navier–Stokes equations, addressed to this line of research, have been found recently by Galanti, Gibbon and Heritage (GGH) [3] and by Gibbon, Fokas and Doering (GFD) [4]. The first authors investigated the mechanism of the vorticity alignment in the Navier–Stokes isotropic turbulence. They started from the results obtained in [5], according to which the vorticity vector $\vec{\omega}$ aligns with the intermediate eigenvector of the strain matrix S . This problem was tackled via the introduction of the variables $\alpha = \frac{\vec{\omega} \cdot S \vec{\omega}}{\vec{\omega} \cdot \vec{\omega}} \equiv \hat{\xi} \cdot S \hat{\xi}$ (the "stretching rate") and $\vec{\chi} = \frac{\vec{\omega} \wedge S \vec{\omega}}{\vec{\omega} \cdot \vec{\omega}} \equiv \hat{\xi} \wedge S \hat{\xi}$ where $\hat{\xi} = \frac{\vec{\omega}}{\omega}$. Two differential equations were derived for α and χ which were exploited to discuss the vorticity alignment in terms of the angle $\phi(x, y, z, t)$ between $\vec{\omega}$ and $S \vec{\omega}$ defined by $\tan \phi = \frac{|\vec{\omega} \wedge S \vec{\omega}|}{\vec{\omega} \cdot S \vec{\omega}} = \frac{\chi}{\alpha}$ ($\chi = \sqrt{\vec{\chi} \cdot \vec{\chi}}$). On the other hand, GFD observed that stretched vortex solutions of the 3D Navier–Stokes equations, such as the Burgers vortices, are characterized by a uni-directional vorticity stretched by a strain field decoupled from them. Then, they showed that these drawbacks can be partially circumvented by searching classes of solutions of the type $\vec{u} = (u_1(x, y, t), u_2(x, y, t), \gamma(x, y, t)z + W(x, y, t))$. In such a way the equations for the third component of vorticity ω_3 and W decouple. Generalizations of the Burgers vortex type are constructed and various solutions for W are discussed.

Following the spirit of the above mentioned works, here we show as vortex solutions of the incompressible 3D Navier–Stokes equations (1.1)–(1.2)

are generated by their symmetry infinitesimal operators. Using a group-theoretical approach, new classes of exact solutions are found. Some of them contain as special cases the Burgers vortex and the shear-layer solutions, which are produced by two different infinitesimal operators. By an algebraic point of view, equations (1.1)–(1.2) are characterized by a symmetry algebra containing arbitrary functions. This fact indicates that this algebra is infinite-dimensional. Subalgebras isomorphic to the algebra of the Euclidean group are present.

An important feature of the technique applied is that the symmetry algebra turns out to be not trivial only if the external force \vec{f} is derivable from a potential function $\varphi(x, y, z, t)$, i.e.

$$\vec{f} = \nabla \varphi. \quad (1.3)$$

The problem of the boundary conditions is also considered. This is a fundamental aspect of the symmetry analysis, because if the boundary conditions are also invariant under the symmetry group, then an invariant solution is the unique solution of the system under investigation [6]. In this way, we have found interesting invariant surfaces which play the role of invariant conditions.

The paper is organized as follows. In Section 2 the infinitesimal operators of the (Lie-point) symmetries admitted by Eqs. (1.1)–(1.2) are obtained, while the corresponding group transformations are described in Section 3. Section 4 is devoted to the determination of the reduced equations coming from the generators of the symmetry subgroups. Some of these equations can be exactly solved and give rise to solutions of the vortex type which generalize well-known configurations as, for example, the Burgers shear-layer and the Burgers vortex. Section 5 deals with the invariance properties of the boundary conditions. Finally, in Section 6 and in Appendices A and B some comments and details of the calculations are reported, respectively.

2 Group analysis

To the aim of finding the symmetry group of Eqs. (1.1)–(1.2), let us introduce the vector field

$$V = \xi_1 \partial_x + \xi_2 \partial_y + \xi_3 \partial_z + \xi_4 \partial_t + \sum_{j=1}^3 \phi_j \partial_{u_j} + \phi_4 \partial_p, \quad (2.1)$$

where ξ_j and ϕ_j are functions of $x, y, z, t, u_1, u_2, u_3, p$, and $\partial_x = \partial/\partial x$, and so on.

We remind the reader that for a given system of differential equations $\Delta_j = 0$, defined on a differentiable manifold M , a local group of transformations G , acting on M , is a group of symmetries of $\Delta_j = 0$ if and only if

$$pr^{(n)}V[\Delta_j] = 0 \quad (2.2)$$

whenever $\Delta_j = 0$, for every generator V of G (see [7], [8] and [9]).

In the case of Eqs. (1.1)–(1.2), the condition (2.2) becomes

$$pr^{(2)}V[\Delta_j] = 0, \quad (2.3)$$

where $pr^{(2)}V$ denotes the second prolongation

$$\begin{aligned} pr^{(2)}V = V + \phi_1^x \partial_{u_{1x}} + \phi_1^y \partial_{u_{1y}} + \phi_1^z \partial_{u_{1z}} + \phi_1^t \partial_{u_{1t}} + \phi_1^{xx} \partial_{u_{1xx}} + \phi_1^{yy} \partial_{u_{1yy}} + \\ \phi_1^{zz} \partial_{u_{1zz}} + \phi_2^x \partial_{u_{2x}} + \phi_2^y \partial_{u_{2y}} + \phi_2^z \partial_{u_{2z}} + \phi_2^t \partial_{u_{2t}} + \phi_2^{xx} \partial_{u_{2xx}} + \phi_2^{yy} \partial_{u_{2yy}} + \\ \phi_2^{zz} \partial_{u_{2zz}} + \phi_3^x \partial_{u_{3x}} + \phi_3^y \partial_{u_{3y}} + \phi_3^z \partial_{u_{3z}} + \phi_3^t \partial_{u_{3t}} + \phi_3^{xx} \partial_{u_{3xx}} + \phi_3^{yy} \partial_{u_{3yy}} + \\ \phi_3^{zz} \partial_{u_{3zz}} + \phi_4^x \partial_{p_x} + \phi_4^y \partial_{p_y} + \phi_4^z \partial_{p_z}, \end{aligned} \quad (2.4)$$

with

$$\phi_j^\alpha = D_\alpha(\phi_j - \sum_{i=1}^4 \xi_i u_i^j) + \sum_{i=1}^4 \xi_i u_{\alpha,i}^j, \quad (2.5)$$

$j = 1, 2, 3, 4$, $u_i^j = \frac{\partial u_j}{\partial x_i}$, $u_{\alpha,i}^j = \frac{\partial u_\alpha^j}{\partial x_i}$, x_i and α stand for x, y, z, t , and x, y, z, t, xx, yy, zz , respectively, D being the derivation operation.

Equation (2.3) provides a set of differential equations, the so-called *determining system*, which allows us to obtain the coefficients ξ_j, ϕ_j . From these we infer the infinitesimal operator generating the symmetry group of Eqs. (1.1)–(1.2), namely

$$V = \sum_{i=1}^9 V_i, \quad (2.6)$$

where

$$V_1 = g\partial_x + \dot{g}\partial_{u_1} + (-\ddot{g}x + gf_1)\partial_p, \quad (2.7)$$

$$V_2 = h\partial_y + \dot{h}\partial_{u_2} + (-\ddot{h}y + hf_2)\partial_p, \quad (2.8)$$

$$V_3 = r\partial_z + \dot{r}\partial_{u_3} + (-\ddot{r}z + rf_3)\partial_p, \quad (2.9)$$

$$V_4 = k\partial_p, \quad (2.10)$$

$$V_5 = a[x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - \sum_{i=1}^3 u_i\partial_{u_i} + (-2p + xf_1 + yf_2 + zf_3 + 2\int^x f_1 dx + 2t\int^x f_{1t} dx)\partial_p], \quad (2.11)$$

$$V_6 = b[y\partial_x - x\partial_y + u_2\partial_{u_1} - u_1\partial_{u_2} + (yf_1 - xf_2)\partial_p], \quad (2.12)$$

$$V_7 = c[z\partial_y - y\partial_z + u_3\partial_{u_2} - u_2\partial_{u_3} + (zf_2 - yf_3)\partial_p], \quad (2.13)$$

$$V_8 = d[z\partial_x - x\partial_z + u_3\partial_{u_1} - u_1\partial_{u_3} + (zf_1 - xf_3)\partial_p], \quad (2.14)$$

$$V_9 = e[\partial_t + (\int^x f_{1t} dx)\partial_p], \quad (2.15)$$

$g(t)$, $h(t)$, $k(t)$, $R(t)$ are arbitrary functions of time of class C^∞ , and a , b , c , d , e arbitrary constants (dots mean time derivatives). The infinitesimal operators (2.7)–(2.15) represent the generators of the Lie–point symmetries of the 3D Navier–Stokes equations (1.1)–(1.2).

3 Group transformations

The integration of the infinitesimal operator (2.6) enables us to find the finite transformations leaving the equations (1.1)–(1.2) invariant. We have that the linear combination $W = V_1 + V_2 + V_3$ gives rise to a transformation to an arbitrary moving coordinate frame of the type

$$G_{\vec{\alpha}} : (\vec{x}, t, \vec{u}, p) \longrightarrow (\vec{x} + \varepsilon \vec{\alpha}, t, \vec{u} + \varepsilon \vec{\alpha}, p - \varepsilon \vec{x} \cdot \vec{\alpha} - \frac{1}{2} \varepsilon^2 \vec{\alpha} \cdot \vec{\alpha} + \int_0^\varepsilon \vec{\alpha} \cdot \vec{f} d\varepsilon) \quad (3.1)$$

where $\vec{\alpha} = (g(t), h(t), r(t))$ and ε is the group parameter.

V_4 is the generator of the pressure change

$$G_p : (\vec{x}, t, \vec{u}, p) \longrightarrow (\vec{x}, t, \vec{u}, p + \varepsilon k(t)), \quad (3.2)$$

while V_5 is the infinitesimal operator for the scale transformations. The vector fields V_6, V_7, V_8 yield both the space and the velocity field rotations. Finally, V_9 produces the time translations together with an adjustment of the pressure, i.e.

$$G : (\vec{x}, t, \vec{u}, p) \longrightarrow (\vec{x}, t + \varepsilon, \vec{u}, p + \varphi(x, y, z, t + \varepsilon) - \varphi(x, y, z, t)), \quad (3.3)$$

where φ is the potential function linked to the external force by (1.3). Then, we can "absorb" \vec{f} in p by re-defining the pressure p as

$$p' = p - \varphi(x, y, z, t).$$

Without introducing new symbols, we shall put formally $\vec{f} = 0$ into Eqs. (1.1)–(1.2) and into the expressions (2.7)–(2.15) for the generators.

4 Reduced equations and exact solutions

By exploiting the generators V_j of the Lie–point transformations (see (2.7)–(2.15)), one can build up exact solutions of Eqs. (1.1)–(1.2) via the symmetry reduction approach. This allows one to lower the order of the system of differential equations under consideration using the invariants associated with a given subgroup of the symmetry group. In the following we present some reductions leading to exact solutions of Eqs. (1.1)–(1.2) of physical interest.

4.1 Case i)

Let us take the vector field V_1 defined by (2.7). A set of basis invariants I_j of the related subgroup can be determined from the finite group transformations

$$\begin{aligned} x' &= x + \varepsilon g(t), \quad y' = y, \quad z' = z, \quad t' = t, \\ u'_1 &= u_1 + \varepsilon \dot{g}, \quad u'_2 = u_2, \quad u'_3 = u_3, \\ p' &= p - \frac{1}{2}\varepsilon \ddot{g} (2x + \varepsilon g). \end{aligned} \quad (4.1)$$

We obtain

$$I_1 = y, \quad I_2 = z, \quad I_3 = t, \quad I_4 = u_1 - \frac{\dot{g}}{g}x, \quad I_5 = u_2, \quad I_6 = u_3, \quad I_7 = p + \frac{\ddot{g}}{2g}x^2. \quad (4.2)$$

By means of the choice of variables

$$\begin{aligned} U_1(y, z, t) &\equiv I_4 = u_1 - \frac{\dot{g}}{g}x, \quad U_2(y, z, t) \equiv I_5 = u_2, \quad U_3(y, z, t) \equiv u_3, \\ \pi(y, z, t) &\equiv I_7 = p + \frac{\ddot{g}}{2g}x^2, \end{aligned} \quad (4.3)$$

the system (1.1)–(1.2) is cast into the reduced form

$$U_{1t} + \frac{\dot{g}}{g}U_1 + U_2U_{1y} + U_3U_{1z} - \nu(U_{1yy} + U_{1zz}) = 0, \quad (4.4a)$$

$$U_{2t} + U_2U_{2y} + U_3U_{2z} + \pi_y - \nu(U_{2yy} + U_{2zz}) = 0, \quad (4.4b)$$

$$U_{3t} + U_2U_{3y} + U_3U_{3z} + \pi_z - \nu(U_{3yy} + U_{3zz}) = 0, \quad (4.4c)$$

$$U_{2y} + U_{3z} + \frac{\dot{g}}{g} = 0. \quad (4.4d)$$

A particular solution of Eqs. (4.4a)–(4.4d) can be found as follows. Let us put [10]

$$U_2 = k_1y, \quad U_3 = \sigma - k_1z, \quad \pi = -k_1^2 \frac{z^2}{2} + k_1\sigma z - k_1^2 \frac{y^2}{2} + \tau(t), \quad (4.5)$$

where k_1, k_2, σ, g are constants, and $\tau(t)$ is an arbitrary function of time. Substitution from (4.5) into Eqs. (4.4a)–(4.4d) yields the linear equation

$$U_{1t} + k_1 y U_{1y} + (\sigma - k_1 z) U_{1z} - \nu (U_{1yy} + U_{1zz}) = 0. \quad (4.6)$$

At this point it is convenient to look for a solution of Eq. (4.6) of the form

$$U_1 = Y(y) T(z) \Phi(t). \quad (4.7)$$

Then, Eq. (4.6) can be written as

$$k_1 y \frac{Y_y}{Y} - \nu \frac{Y_{yy}}{Y} = (k_1 z - \sigma) \frac{T_z}{T} + \nu \frac{T_{zz}}{T} - \frac{\Phi_t}{\Phi} \equiv G, \quad (4.8)$$

G being an arbitrary constant. On the other hand, Eq. (4.8) tells us that

$$(k_1 z - \sigma) \frac{T_z}{T} + \nu \frac{T_{zz}}{T} = G + \frac{\Phi_t}{\Phi} \equiv H,$$

where H is an arbitrary constant. To summarize, the functions Y, T, Φ obey the ordinary differential equations

$$\Phi_t = (H - G) \Phi, \quad (4.9a)$$

$$\nu Y_{yy} - k_1 y Y_y + G Y = 0, \quad (4.9b)$$

$$\nu T_{zz} + (k_1 z - \sigma) T_z - H T = 0. \quad (4.9c)$$

Equations (4.9a)–(4.9c) afford, respectively, the general solutions

$$\Phi = c_1 \exp[(H - G)t], \quad (4.10a)$$

$$Y = c_2 M\left(-\frac{G}{2k_1}, \frac{1}{2}, \frac{k_1 y^2}{2\nu}\right) + y c_3 M\left(\frac{1}{2} - \frac{G}{2k_1}, \frac{3}{2}, \frac{k_1 y^2}{2\nu}\right), \quad (4.10b)$$

$$T = c_4 M\left(-\frac{H}{2k_1}, \frac{1}{2}, \frac{-k_1(z - \frac{\sigma}{k_1})^2}{2\nu}\right) + (z - \frac{\sigma}{k_1}) c_5 M\left(\frac{1}{2} - \frac{H}{2k_1}, \frac{3}{2}, \frac{-k_1(z - \frac{\sigma}{k_1})^2}{2\nu}\right), \quad (4.10c)$$

with c_1, \dots, c_5 arbitrary constants. M is the Kummer function defined by [7]

$$M(\alpha, \beta, z) = 1 + \frac{\alpha}{\beta}z + \frac{(\alpha)_2}{(\beta)_2} \frac{z^2}{2!} + \dots + \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!} + \dots, \quad (4.4)$$

with

$$(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2)\dots(\alpha + n - 1), \quad (\alpha)_0 = 1. \quad (4.5)$$

Consequently, from (4.3) we obtain

$$\begin{aligned} u_1 = c_1 e^{(H-G)t} \{ & [c_2 M\left(-\frac{G}{2k_1}, \frac{1}{2}, \frac{k_1 y^2}{2\nu}\right) + y c_3 M\left(\frac{1}{2} - \frac{G}{2k_1}, \frac{3}{2}, \frac{k_1 y^2}{2\nu}\right)] \times \\ & [c_4 M\left(-\frac{H}{2k_1}, \frac{1}{2}, \frac{-k_1(z - \frac{\sigma}{k_1})^2}{2\nu}\right) + (z - \frac{\sigma}{k_1}) c_5 M\left(\frac{1}{2} - \frac{H}{2k_1}, \frac{3}{2}, \frac{-k_1(z - \frac{\sigma}{k_1})^2}{2\nu}\right)] \}, \end{aligned} \quad (4.11a)$$

$$u_2 = k_1 y, \quad (4.11b)$$

$$u_3 = \sigma - k_1 z, \quad (4.11c)$$

$$p = -k_1^2 \frac{z^2}{2} + k_1 \sigma z + \tau(t) - k_1^2 \frac{y^2}{2}. \quad (4.11d)$$

It is noteworthy that at least in some special cases, Eqs. (4.11a)–(4.11d) reproduce interesting solutions which can be interpreted by a physical point of view, such as the Burgers shear-layer and other solutions. This aspect is discussed below.

4.1.1 The Burgers shear-layer and other solutions

We observe that Eqs. (4.11a)–(4.11d) give rise to static solutions for $H = G \equiv \lambda$. We shall distinguish two cases: *a*) $\lambda = 0$, and *b*) $\lambda \neq 0$.

Case a)

From (4.1a) we obtain

$$u_1 = [c_2 + c_3 \sqrt{-\frac{\pi\nu}{k_1}} \operatorname{erf}(\sqrt{-\frac{k_1}{2\nu}}y)][c_4 + c_5 \sqrt{\frac{\pi\nu}{k_1}} \operatorname{erf}(\sqrt{\frac{k_1}{2\nu}}z)], \quad (4.12)$$

where for simplicity $\sigma \equiv 0$ and we have used the properties $M(0, b, \varsigma) = 1$ and [11, p. 509]]

$$M(\frac{1}{2}, \frac{3}{2}, -\varsigma^2) = \frac{\sqrt{\pi}}{2\varsigma} \operatorname{erf} \varsigma,$$

with

$$\operatorname{erf} \varsigma = \frac{2}{\sqrt{\pi}} \int_0^\varsigma \exp(-\theta^2) d\theta$$

We remark that by choosing $k_1 = -\gamma$ ($\gamma > 0$) and $c_5 = 0$ in Eq.(4.12), we get the Burgers shear-layer solution [3, 1]

$$\vec{u} = (u_1(y), -\gamma y, \gamma z)^T, \quad (4.13)$$

where

$$Y(y) \equiv u_1(y) = A \sqrt{\frac{\pi\nu}{\gamma}} \operatorname{erf}(\sqrt{\frac{\gamma}{2\nu}}y) + B, \quad (4.14)$$

with A and B arbitrary constants.

Now let us recall some basic quantities appearing in the study of turbulent flows. They are the vorticity $\vec{\omega} = \nabla \wedge \vec{u}$, the strain matrix S whose elements are defined by $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, where $u_{i,j} \equiv \frac{\partial u_i}{\partial x_j}$ and $u_{j,i} \equiv \frac{\partial u_j}{\partial x_i}$, the energy dissipation $S_{ij}S_{ij}$ and the enstrophy $\omega_i\omega_i$ (the summation under repeated indices is understood) [11]–[12]. Furthermore, in the study of the vortex alignments, it is important the dynamic angle $\phi(x, y, z, t)$ defined in the Introduction.

The vorticity corresponding to the solution (4.14) is $\vec{\omega} = (0, 0, \omega_3)^T$ with

$$\omega_3 = -Y'(y) = -A \sqrt{\frac{2\nu}{\gamma}} \exp(-\frac{\gamma y^2}{2\nu}), \quad (4.15)$$

while the strain matrix S is given by

$$S = \begin{pmatrix} 0 & \frac{1}{2}Y' & 0 \\ \frac{1}{2}Y' & -\gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \quad (4.16)$$

The energy dissipation reads

$$S_{ij}S_{ij} = 2\gamma^2 + \frac{1}{2}Y'^2, \quad (4.17)$$

which tells us that the dissipation is high when the vorticity is so [4]. We observe that the vorticity vanishes for large values of y^2 . We notice also that the pressure p satisfies the equation

$$\frac{\partial p}{\partial z} = -\gamma z,$$

which is just the equation (66) shown in [3]. Within our framework, p arises as a consequence of the reduction procedure involving the generator V_1 . The dynamic angle ϕ related to the vortex solution (4.13) turns out to be zero, so that the vectors $\vec{\omega}$ and $S \vec{\omega}$ are parallel.

From formula (4.12) we deduce that for $k_1 > 0$, by switching off the constant c_3 a shear-layer of the Burgers type of the form $\vec{u} = (Z(z), -\gamma y, \gamma z)^T$ emerges, where now

$$Z(z) \equiv u_1(z) = A \sqrt{\frac{\pi\nu}{k_1}} \operatorname{erf}(\sqrt{\frac{k_1}{2\nu}} z) + B. \quad (4.18)$$

Therefore, (4.12) can be interpreted as a more general vortex configuration of the Burgers shear-layer type in which both the variables y and z are involved. *Case b)*

Formula (4.11a) can be exploited to find other interesting static solutions which can be considered as generalizations of the Burgers shear-layer. An explicit example corresponds to the choice $\lambda = -k_1$ and $\sigma = c_3 = c_5 = 0$. We have

$$u_1 = A \exp\left[\frac{k_1}{2\nu}(y^2 - z^2)\right], \quad (4.19a)$$

$$u_2 = k_1 y, \quad (4.19b)$$

$$u_3 = -k_1 z, \quad (4.19c)$$

A being a constant.

The vorticity $\vec{\omega}$ has $\omega_1 = 0$ and the remaining components different from zero. Precisely

$$\omega_2 = -\frac{Ak_1}{\nu} z \exp\left[\frac{k_1}{2\nu}(y^2 - z^2)\right], \quad (4.20a)$$

$$\omega_3 = -\frac{Ak_1}{\nu} y \exp\left[\frac{k_1}{2\nu}(y^2 - z^2)\right]. \quad (4.20b)$$

A possible physical interpretation of this situation is that for $k_1 < 0$, it represents a vortex structure where ω_2 behaves as a shear-layer of the Burgers type along the axis y for any fixed value of z , while in opposition to ω_2 , ω_3 vanishes at $y = 0$. The evaluation of the dynamic angle for the vortex (4.19a)–(4.19c) leads to the formula

$$\tan \phi = \frac{2xy}{y^2 - z^2} \equiv \sinh 2\theta, \quad (4.21)$$

where we have put $y = \cosh \theta$, $z = \sinh \theta$. For example, we have $\phi = 0$ for $\theta = 0$, and $\phi = \pm\frac{\pi}{2}$ for $\theta \rightarrow \pm\infty$.

An example of a non-static solution

Non-static solutions can be obtained from (4.11a)–(4.11d) assuming that $H - G \neq 0$. An interesting example is given by

$$u_1 = A \exp\left(-\frac{\gamma}{2}t\right) \exp\left(-\frac{\gamma y^2}{4\nu}\right) y^{\frac{1}{2}} I_{-\frac{1}{4}}\left(\frac{\gamma y^2}{4\nu}\right), \quad (4.22a)$$

$$u_2 = -\gamma y, \quad (4.22b)$$

$$u_3 = \gamma z, \quad (4.22c)$$

$$p = -\frac{\gamma^2}{2}(y^2 + z^2), \quad (4.22d)$$

which is derived from (4.11a)–(4.11d) for $H = 0, G = \frac{\gamma}{2}, k_1 = -\gamma (\gamma > 0), c_3 = 0, c_5 = 0, \sigma = 0, \tau = 0$, where I denotes the modified Bessel function.

The vorticity is $\vec{\omega} = (0, 0, \omega_3)$, with

$$\omega_3 = -u_{1y} = B \exp(-\frac{\gamma}{2}t) \exp(-\frac{\gamma y^2}{4\nu}) y^{\frac{3}{2}} [I_{-\frac{1}{4}}(\frac{\gamma y^2}{4\nu}) - I_{\frac{3}{4}}(\frac{\gamma y^2}{4\nu})]. \quad (4.23)$$

The strain matrix and the dissipation corresponding to the solution (4.22a)–(4.22d) can be derived from (4.16) and (4.17), respectively, via the substitution of Y' by u_{1y} (see (4.13)). When ν and γ are fixed, for large y we have

$$\omega_3 \sim y^{-\frac{3}{2}},$$

while the total strain behaves as

$$S_{ij}S_{ij} \sim 2\gamma^2 + \frac{A^2\nu}{8\pi\gamma} y^{-3}. \quad (4.24)$$

The dynamic angle corresponding to the solution (4.40)–(4.42) is zero. Therefore, this vortex-like structure is characterized by a vorticity which is aligned with $S \vec{\omega}$.

4.2 Case ii)

The vector field

$$V_2 = h\partial_y + \dot{h}\partial_{u_2} - \ddot{h}y\partial_p \quad (4.25)$$

is related to the invariants

$$t, x, z, u_2 - \frac{\dot{h}}{h}y, u_1, u_3. \quad (4.26)$$

Then, by using the variables

$$u_1 = U_1(x, z, t), \quad (4.27a)$$

$$u_2 = U_2(x, z, t) + \frac{\dot{h}}{h}y, \quad (4.27b)$$

$$u_3 = U_3(x, z, t), \quad (4.27c)$$

$$p = \pi - \frac{\ddot{h}}{2h}y^2, \quad (4.27d)$$

we are led to the reduced system

$$U_{1t} + U_1 U_{1x} + U_3 U_{1z} + \pi_x - \nu(U_{1xx} + U_{1zz}) = 0, \quad (4.28a)$$

$$U_{2t} + U_1 U_{2x} + U_2 \frac{\dot{h}}{h} + U_3 U_{2z} - \nu(U_{2xx} + U_{2zz}) = 0, \quad (4.28b)$$

$$U_{3t} + U_1 U_{3x} + U_3 U_{3z} + \pi_z - \nu(U_{3xx} + U_{3zz}) = 0, \quad (4.28c)$$

$$U_{1x} + U_{3z} + \frac{\dot{h}}{h} = 0. \quad (4.28d)$$

By changing x with y and U_2 with U_1 , this system can be discussed in a way similar to that followed for Eqs. (4.4a)–(4.4d). Of course, starting from the field V_3 a reduced system analogous to Eqs. (4.28a)–(4.28d) is derived.

4.3 Case iii)

Let us deal with the vector field

$$\begin{aligned} W = V_1 + V_2 + V_3 = & g\partial_x + h\partial_y + r\partial_z + \dot{g}\partial_{u_1} + \dot{h}\partial_{u_2} + \dot{r}\partial_{u_3} \\ & + (\ddot{g}x + \ddot{h}y + \ddot{r}z)\partial_p. \end{aligned} \quad (4.29)$$

The group transformations associated with W are

$$\begin{aligned} x' &= x + \varepsilon g, y' = y + \varepsilon h, z' = z + \varepsilon r, t' = t, \\ u'_1 &= u_1 + \varepsilon \dot{g}, u'_2 = u_2 + \varepsilon \dot{h}, u'_3 = u_3 + \varepsilon \dot{r}, \\ p' &= p - \frac{\varepsilon \ddot{g}}{2}(2x + \varepsilon g) - \frac{\varepsilon \ddot{h}}{2}(2y + \varepsilon h) - \frac{\varepsilon \ddot{r}}{2}(2z + \varepsilon r) \end{aligned} \quad (4.30)$$

From (4.30) we deduce the invariants

$$I_o = t, I_1 = u_1 - x\frac{\dot{g}}{g}, I_2 = u_2 - y\frac{\dot{h}}{h}, I_3 = z\frac{\dot{r}}{r}, \quad (4.6)$$

$$I_4 = p + \frac{1}{2}x^2 \frac{\dot{g}}{g} + \frac{1}{2}y^2 \frac{\dot{h}}{h} + \frac{1}{2}z^2 \frac{\dot{r}}{r}. \quad (4.31)$$

Now let us introduce the variables

$$U_1 = I_1, U_2 = I_2, U_3 = I_3, \pi = I_4, \quad (4.32)$$

depending on the (invariant) $I_o = t$ only. Then, Eqs. (1.1)–(1.2) furnish the reduced system

$$U_{1t} + \frac{\dot{g}}{g} U_1 = 0, \quad (4.33a)$$

$$U_{2t} + \frac{\dot{h}}{h} U_2 = 0, \quad (4.33b)$$

$$U_{3t} + \frac{\dot{r}}{r} U_3 = 0, \quad (4.33c)$$

$$\frac{\dot{g}}{g} + \frac{\dot{h}}{h} + \frac{\dot{r}}{r} = 0. \quad (4.33d)$$

These equations give

$$U_1 = \frac{c_1}{g}, U_2 = \frac{c_2}{h}, U_3 = \frac{c_3}{r}, \quad (4.34)$$

and

$$U_1 U_2 U_3 = \text{const.} \quad (4.35)$$

Then, from (4.31) and (4.34) we infer

$$u_1 = \frac{c_1}{g} + x \frac{\dot{g}}{g}, u_2 = \frac{c_2}{h} + y \frac{\dot{h}}{h}, u_3 = \frac{c_3}{r} + z \frac{\dot{r}}{r}, \quad (4.36)$$

while the condition (4.33d) ensures that the equation $\nabla \cdot \vec{u} = 0$ is satisfied. The solutions (4.36) imply that the vorticity $\vec{\omega} = \nabla \wedge \vec{u}$ vanishes, so that the motion is irrotational. Thus, we may introduce the velocity potential ϕ defined by $\vec{u} = \nabla \phi$. We have

$$\phi = \frac{1}{2} \left(x^2 \frac{\dot{g}}{g} + y^2 \frac{\dot{h}}{h} + z^2 \frac{\dot{r}}{r} \right) + \frac{c_1}{g} x + \frac{c_2}{h} y + \frac{c_3}{r} z + \phi_o. \quad (4.37)$$

Of course, ϕ fulfills the Laplace equation $\nabla^2 \phi = 0$.

4.4 Case iv)

The generator V_5 gives rise to the finite group transformations

$$x' = x \exp(\epsilon), y' = y \exp(\epsilon), z' = z \exp(\epsilon), t' = t \exp(2\epsilon), u_1' = u_1 \exp(-\epsilon),$$

$$u_2' = u_2 \exp(-\epsilon), u_3' = u_3 \exp(-\epsilon), p' = p \exp(-2\epsilon). \quad (4.38)$$

A set of basis invariants is

$$\xi = \frac{x}{y}, \eta = \frac{x}{z}, \theta = \frac{x^2}{t}, \Lambda_1(\xi, \eta, \theta) = u_1 x, \Lambda_2(\xi, \eta, \theta) = u_2 x,$$

$$\Lambda_3(\xi, \eta, \theta) = u_3 x, \pi(\xi, \eta, \theta) = px^2. \quad (4.39)$$

The reduced equations read

$$\begin{aligned} & -2\pi + 2\theta\pi_\theta + \eta\pi_\eta + \xi\pi_\xi - 2\nu\Lambda_1 - \Lambda_1^2 + (2\nu - \theta)\theta\Lambda_{1\theta} + 2\theta\Lambda_1\Lambda_{1\theta} - 4\theta^2\nu\Lambda_{1\theta\theta} + \\ & 2\nu\eta(1 - \eta^2)\Lambda_{1\eta} + \eta\Lambda_1\Lambda_{1\eta} - \eta^2\Lambda_3\Lambda_{1\eta} - 4\nu\eta\theta\Lambda_{1\eta\theta} - \nu\eta^2(1 + \eta^2)\Lambda_{1\eta\eta} + 2\nu\xi(1 - \xi^2)\Lambda_{1\xi} + \\ & \xi\Lambda_1\Lambda_{1\xi} - \xi^2\Lambda_2\Lambda_{1\xi} - 4\nu\xi\theta\Lambda_{1\xi\theta} - 2\nu\xi\eta\Lambda_{1\xi\eta} - \nu\xi^2(1 + \xi^2)\Lambda_{1\xi\xi} = 0, \quad (4.40a) \end{aligned}$$

$$\begin{aligned} & -2\nu\Lambda_2 - \Lambda_1\Lambda_2 + (2\nu - \theta)\theta\Lambda_{2\theta} + 2\theta\Lambda_1\Lambda_{2\theta} - 4\theta^2\nu\Lambda_{2\theta\theta} + \\ & 2\nu\eta(1 - \eta^2)\Lambda_{2\eta} + \eta\Lambda_1\Lambda_{2\eta} - \eta^2\Lambda_3\Lambda_{2\eta} - 4\nu\eta\theta\Lambda_{2\eta\theta} - \nu\eta^2(1 + \eta^2)\Lambda_{2\eta\eta} + 2\nu\xi(1 - \xi^2)\Lambda_{2\xi} + \\ & \xi\Lambda_1\Lambda_{2\xi} - \xi^2\Lambda_2\Lambda_{2\xi} - 4\nu\xi\theta\Lambda_{2\xi\theta} - 2\nu\xi\eta\Lambda_{2\xi\eta} - \nu\xi^2(1 + \xi^2)\Lambda_{2\xi\xi} - \xi^2\pi_\xi = 0, \quad (4.40b) \end{aligned}$$

$$\begin{aligned} & -2\nu\Lambda_3 - \Lambda_1\Lambda_3 + (2\nu - \theta)\theta\Lambda_{3\theta} + 2\theta\Lambda_1\Lambda_{3\theta} - 4\theta^2\nu\Lambda_{3\theta\theta} + \\ & 2\nu\eta(1 - \eta^2)\Lambda_{3\eta} + \eta\Lambda_1\Lambda_{3\eta} - \eta^2\Lambda_3\Lambda_{3\eta} - 4\nu\eta\theta\Lambda_{3\eta\theta} - \nu\eta^2(1 + \eta^2)\Lambda_{3\eta\eta} + 2\nu\xi(1 - \xi^2)\Lambda_{3\xi} + \\ & \xi\Lambda_1\Lambda_{3\xi} - \xi^2\Lambda_2\Lambda_{3\xi} - 4\nu\xi\theta\Lambda_{3\xi\theta} - 2\nu\xi\eta\Lambda_{3\xi\eta} - \nu\xi^2(1 + \xi^2)\Lambda_{3\xi\xi} - \xi^2\pi_\xi = 0, \quad (4.40c) \end{aligned}$$

$$\xi\Lambda_1\Lambda_{3\xi} - \xi^2\Lambda_2\Lambda_{3\xi} - 4\nu\xi\theta\Lambda_{3\xi\theta} - 2\nu\xi\eta\Lambda_{3\xi\eta} - \nu\xi^2(1+\xi^2)\Lambda_{3\xi\xi} - \eta^2\pi_\eta = 0, \quad (4.40c)$$

$$-\Lambda_1 + 2\theta\Lambda_{1\theta} + \eta\Lambda_{1\eta} + \xi\Lambda_{1\xi} - \eta^2\Lambda_{3\eta} - \xi^2\Lambda_{2\xi} = 0. \quad (4.40d)$$

A special solution of this huge nonlinear system of partial differential equations is derived in Appendix B. It reads

$$u_1 = ct^{-\frac{1}{2}}, \quad (4.41a)$$

$$u_2 = -2at^{-\frac{1}{2}} + c_1(\nu t^{-1})^{\frac{1}{2}} \exp\left[\frac{xt^{-\frac{1}{2}}}{4\nu}(4c - xt^{-\frac{1}{2}})\right] - c_2(\nu t^{-1})^{\frac{1}{2}} \exp\left[\frac{xt^{-\frac{1}{2}}}{4\nu}(4c - xt^{-\frac{1}{2}})\right] \times \operatorname{erf} i\left[\frac{1}{2}\nu^{-\frac{1}{2}}(2c - xt^{-\frac{1}{2}})\right], \quad (4.41b)$$

$$u_3 = -2bt^{-\frac{1}{2}} + c_3(\nu t^{-1})^{\frac{1}{2}} \exp\left[\frac{xt^{-\frac{1}{2}}}{4\nu}(4c - xt^{-\frac{1}{2}})\right] - c_4(\nu t^{-1})^{\frac{1}{2}} \exp\left[\frac{xt^{-\frac{1}{2}}}{4\nu}(4c - xt^{-\frac{1}{2}})\right] \times \operatorname{erf} i\left[\frac{1}{2}\nu^{-\frac{1}{2}}(2c - xt^{-\frac{1}{2}})\right], \quad (4.41c)$$

$$p = c_o t^{-1} + t^{-\frac{3}{2}}\left(\frac{c}{2}x - ay - bz\right), \quad (4.41d)$$

where

$$\operatorname{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^\zeta \exp(-t^2) dt, \quad (4.42)$$

$$\operatorname{erf} i(\zeta) = -i \operatorname{erf}(i\zeta) = \frac{-2i}{\sqrt{\pi}} \int_0^{i\zeta} \exp(-t^2) dt, \quad (4.43)$$

and a, b, c and c_j ($j = 0, 1, 2, 3, 4$) are constants.

The vorticity corresponding to the solution (4.41a)–(4.41d) is

$$\vec{\omega} = \left(0, -\frac{2}{t}\Lambda_{3\theta}, -\frac{1}{x^2}\Lambda_2 + \frac{2}{t}\Lambda_{2\theta}\right)^T, \quad (4.44)$$

where $\Lambda_2(\theta)$ and $\Lambda_3(\theta)$ are expressed in Appendix B. We notice that in this case we obtain a stretching rate coinciding with the null vector, i.e. $S \vec{\omega} \equiv (0, 0, 0)^T$, so that the dynamic angle $\phi = \arctan(\frac{\chi}{\alpha})$ is not definite.

4.5 Case v)

Let us consider the symmetry operator V_6 . The corresponding group transformations are

$$x' = x \cos \varepsilon + y \sin \varepsilon, \quad (4.45a)$$

$$y' = -x \sin \varepsilon + y \cos \varepsilon, \quad (4.45b)$$

$$z' = z, \quad (4.45c)$$

$$t' = t, \quad (4.45d)$$

$$U_1 = u_1 \cos \varepsilon + u_2 \sin \varepsilon, \quad (4.45e)$$

$$U_2 = -u_1 \sin \varepsilon + u_2 \cos \varepsilon, \quad (4.45f)$$

$$U_3 = u_3, \quad (4.45g)$$

$$p' = p \quad (4.45h)$$

which leave the quantities

$$z, t, r = (x^2 + y^2)^{\frac{1}{2}}, \quad (4.46)$$

$$\pi = p \quad (4.47)$$

invariant.

Putting

$$x = r \cos \varepsilon, y = r \sin \varepsilon, \quad (4.48a)$$

$$u_1 = U_1 \cos \varepsilon - U_2 \sin \varepsilon, \quad (4.48b)$$

$$u_2 = U_1 \sin \varepsilon + U_2 \cos \varepsilon, \quad (4.48c)$$

$$u_3 = U_3, \quad (4.48d)$$

where $U_j = U_j(r, z, t)$, Eqs. (1.1)–(1.2) take the form

$$U_{1t} + U_1 U_{1r} - \frac{U_2^2}{r} + U_{1z} U_3 + \pi_r - \nu(U_{1rr} + \frac{U_{1r}}{r} - \frac{U_1}{r^2} + U_{1zz}) = 0, \quad (4.49)$$

$$U_{2t} + U_1 U_{2r} + \frac{U_1 U_2}{r} + U_{2z} U_3 - \nu(U_{2rr} + \frac{U_{2r}}{r} - \frac{U_2}{r^2} + U_{2zz}) = 0, \quad (4.50)$$

$$U_{1r} + \frac{U_1}{r} + U_{3z} = 0, \quad (4.51)$$

$$U_{3t} + U_1 U_{3r} + U_3 U_{3z} - \nu(U_{3rr} + \frac{U_{3r}}{r} + U_{3zz}) + \pi_z = 0. \quad (4.52)$$

To provide an example of exact solution to the reduced system (4.49)–(4.52), let us look for a solution such that

$$U_1 = \frac{\alpha_o}{r}, \quad (4.53a)$$

$$U_2 = \frac{\beta_o}{r}, \quad (4.53b)$$

where α_o, β_o are constants. Then, Eq. (4.49) yields

$$\pi_r = \frac{\alpha_o - \beta_o}{r^3}, \quad (4.53c)$$

while Eq. (4.50) turns out to be identically satisfied. Furthermore, Eq. (4.51) tells us that $U_{3z} = 0$, i.e. $U_3 = U_3(r, t)$. Consequently, Eq. (4.52) becomes

$$U_{3t} + \frac{\alpha_o - \nu}{r} U_{3r} - \nu U_{3rr} = -\pi_z. \quad (4.54)$$

The compatibility condition $\pi_{rz} = \pi_{zr}$ gives

$$\partial_r(U_{3t} + \frac{\alpha_o - \nu}{r} U_{3r} - \nu U_{3rr}) = 0. \quad (4.55)$$

By integrating (4.53c) with respect to r we get

$$\pi = -\frac{\alpha_o^2 + \beta_o^2}{2r^2} + F(z, t), \quad (4.56)$$

$F(z, t)$ being a function of integration. Since $\pi_z = F_z$, we deduce that $F(z, t)$ has to be of the form

$$F(z, t) = a(t)z + b(t), \quad (4.57)$$

where a, b are function of time. Hence

$$U_{3t} + \frac{\alpha_o - \nu}{r} U_{3r} - \nu U_{3rr} = -a_o, \quad (4.58)$$

where we have supposed $a = a_o = \text{const}$, $b = b_o = \text{const}$. From (4.56) we obtain the expression for the pressure p , which reads

$$p \equiv \pi = -\frac{\alpha_o^2 + \beta_o^2}{2r^2} + a_o z + b_o. \quad (4.59)$$

To find U_3 , we shall distinguish two cases. Precisely:

I) $\alpha_o = \nu$.

Then, Eq. (4.58) takes the form of the heat equation with a constant source, namely

$$U_{3t} - \nu U_{3rr} = -a_o. \quad (4.60)$$

A solution of Eq. (4.60) is given by

$$U_3 = \frac{\gamma_o}{\sqrt{\nu t}} \exp\left(-\frac{r^2}{4\nu t}\right) + \frac{a_o}{2\nu} r^2, \quad (4.61)$$

where γ_o is a constant. Thus, collecting all the information, we find that the Navier–Stokes equations (1.1)–(1.2) afford the solution

$$u_1 = \frac{\nu x - \beta_o y}{r^2}, \quad (4.62a)$$

$$u_2 = \frac{\beta_o x + \nu y}{r^2}, \quad (4.62b)$$

$u_3 = U_3$ (see (4.48d)), and (see (4.61))

$$p = -\frac{\nu^2 + \beta_o^2}{2r^2} + a_o z + b_o. \quad (4.62c)$$

II) $\alpha_o \neq \nu$.

In this case Eq. (4.58) can be transformed into the equation

$$\Psi_t + \frac{\alpha_o - \nu}{r} \Psi_r - \nu \Psi_{rr} = 0 \quad (4.63)$$

via the change of variable

$$U_3 = \Psi(r, t) - a_o t. \quad (4.64)$$

By setting

$$\Psi = M(t) N(r), \quad (4.65)$$

Eq. (4.63) is splitted into the ordinary differential equations

$$M_t = -\delta M, \quad (4.66a)$$

$$\nu N_{rr} - \frac{\alpha_o - \nu}{r} N_r + \delta N = 0, \quad (4.66b)$$

where $\delta > 0$ is a constant. These equations can be easily solved to give

$$M = M_o \exp(-\delta t), \quad (4.67a)$$

$$N = r^\mu [c_1 J_\mu(\sqrt{\frac{\delta}{\nu}} r) + c_2 Y_\mu(\sqrt{\frac{\delta}{\nu}} r)], \quad (4.67b)$$

where M_o, c_1, c_2 are constant, $\mu = \frac{\alpha_o}{2\nu}$, and J_μ, Y_μ denote the Bessel functions of the first and the second kind, respectively. Substituting (4.67a) and (4.67b) into (4.64) provides

$$u_3 = U_3 = M_o \exp(-\delta t) r^\mu [c_1 J_\mu(\sqrt{\frac{\delta}{\nu}} r) + c_2 Y_\mu(\sqrt{\frac{\delta}{\nu}} r)] - a_o t. \quad (4.68)$$

The components u_1, u_2 and the pression p are

$$u_1 = \frac{\alpha_o x - \beta_o y}{r^2}, \quad (4.69a)$$

$$u_2 = \frac{\alpha_o y + \beta_o x}{r^2}, \quad (4.69b)$$

$$p = -\frac{\alpha_o^2 + \beta_o^2}{2r^2} + a_o z + b_o, \quad (4.69c)$$

respectively.

4.5.1 The Burgers vortex and other solutions

It is noteworthy that the infinitesimal operator V_6 leads to the Burgers vortex solution [14] as a special case, namely:

$$u_1 = -\frac{\gamma}{2} x - y f(r) \quad (4.70a)$$

$$u_2 = -\frac{\gamma}{2} y + x f(r) \quad (4.70b)$$

$$u_3 = \gamma z, \quad (4.70c)$$

where

$$f(r) = f_0 \frac{1 - \exp(-ar^2)}{r^2}, \quad (4.71)$$

γ and f_0 are constants, and

$$a = \frac{\gamma}{4\nu}. \quad (4.72)$$

In fact, by choosing

$$U_1 = -\frac{\gamma}{2} r, \quad U_2 = r f(r), \quad U_3 = \gamma z, \quad (4.73)$$

the reduced equations (4.49)–(4.52) are satisfied provided that

$$\pi_r = r f^2(r) - \frac{\gamma^2}{4} r, \quad \pi_z = -\gamma^2 z, \quad (4.74)$$

where $f(r)$ obeys the equation

$$f_{rr} + \left(\frac{\gamma}{2\nu} r + \frac{3}{r} \right) f_r + \frac{\gamma}{\nu} f = 0. \quad (4.75)$$

The general solution of Eq (4.75) can be written as

$$f(r) = \frac{f_0 - (f_0 - f_1) \exp(-ar^2)}{r^2}, \quad (4.76)$$

which reproduces just (4.71) for $f_1 = 0$. Eqs. (4.70a)–(4.70c) are derived from (4.49)–(4.52), with the help of (4.73).

Eqs. (4.74) give the pressure ($p = \pi$) :

$$p = -\frac{\gamma^2}{2} \left(z^2 + \frac{r^2}{4} \right) - \frac{f_0^2}{2r^2} \left[1 - \exp(-ar^2) \right]^2 + af_0^2 \left[\text{Ei}(-ar^2) - \text{Ei}(-2ar^2) \right], \quad (4.77)$$

where Ei denotes the exponential–integral function [11]

$$\text{Ei}(\zeta) = - \int_{-\zeta}^{\infty} \frac{\exp(-t)}{t} dt,$$

while the vorticity is

$$\vec{\omega} = (0, 0, \omega_3)^T \quad (4.78)$$

with

$$\omega_3 = 2f + r f_r = 2a f_0 \exp(-ar^2). \quad (4.79)$$

We point out that (4.79) coincides with the expression for the Burgers vortex [14], in which there exists a balance of dissipation and stretching [3], [15], [4].

Now by integrating Eqs. (4.70a) and (4.70c) we easily find

$$r = r_0 \exp\left(-\frac{\gamma}{2}t\right), \quad (4.80)$$

which tells us that in the limit $t \rightarrow \infty$, $r \rightarrow 0$. For $r \rightarrow 0$ both $f(r)$ given by (4.71) and the vorticity (4.79) turn out to be finite.

4.5.2 The Burgers-Lundgren solution

Another interesting solution of the Navier-Stokes equations (1.1)–(1.2) of the Burgers vortex type, generated by the vector field V_6 , is obtained assuming that

$$U_1 = -\frac{\gamma}{2}r, \quad U_2 = r f(r, t), \quad U_3 = \gamma z, \quad (4.81)$$

where the function $f(r, t)$ depends on the time also.

In doing so, substitution from (4.139) into Eqs. (4.49)-(4.52) gets

$$f_t - \nu f_{rr} - \left(\frac{\gamma}{2} r + \frac{3\nu}{r} \right) f_r - \gamma f = 0. \quad (4.82)$$

By inspection, Eq. (4.82) is satisfied by

$$f(r, t) = f_0 \frac{1 - \exp \left[-ar^2 \left(\frac{1}{1 - \exp(-\gamma t)} \right) \right]}{r^2}. \quad (4.83)$$

In this case, the vorticity is time dependent and takes the form

$$\vec{\omega} = (0, 0, \omega_3(r, t))^T,$$

with

$$\omega_3(r, t) = 2f(r, t) + rf_r(r, t) = 2a f_0 \frac{\exp \left(1 - \frac{ar^2}{1 - \exp(-\gamma t)} \right)}{1 - \exp(-\gamma t)}. \quad (4.84)$$

We remark that this quantity is the same as the expression of the vorticity given by Lundgren [15, formula (16)]. When $t \rightarrow \infty$, $\omega_3(r, t)$ tends to the Burgers vortex (4.71). Therefore, as it was noted in [15] one can resort to the assumption of Townsend [16], who considered the axial strain rate proportional to the root-mean-square strain rate in turbulent flow. Under this hypothesis, the Burgers vortex leads naturally to the Kolmogorov length

$$\eta = \left(\frac{\nu^3}{\epsilon} \right)^{1/4},$$

where ϵ is the dissipation rate per unit mass.

4.5.3 Another vortex-like solution of Eq. (4.82)

Let us look for a solution of Eq. (4.82) of the form

$$f = A(t) \exp[B(t) r^2], \quad (4.85)$$

where A and B are functions of time to be determined. Inserting (4.85) into Eq. (4.82) yields

$$A = \frac{A_0}{[\exp \frac{\gamma}{2}t - \frac{c_1}{c_0\gamma} \exp(-\frac{\gamma}{2}t)]^2},$$

$$B = -\frac{a}{[1 - \frac{c_1}{c_0\gamma} \exp(-\frac{\gamma}{2}t)]},$$

with $a = \frac{\gamma}{4\nu}$.

By choosing $\frac{c_1}{c_0\gamma} = 1$, we obtain

$$f(r, t) = \frac{f_0}{2} \sec h^2 \left(\frac{\gamma}{2}t \right) \exp \left(-\frac{ar^2}{1 + \exp(-\gamma t)} \right). \quad (4.86)$$

In opposition to the behavior of (4.83) and (4.84), this solution tends to zero when $t \rightarrow \infty$.

4.5.4 Evaluation of the dynamic angles for vortices related to V_6

1) The solution (4.61)-(4.62b) has a vorticity given by $\vec{\omega} = (\omega_1, \omega_2, 0)^T$, with

$$\omega_1 = [\frac{a_o}{\nu} - \frac{\gamma_o}{2}(\nu t)^{-\frac{3}{2}} \exp(-\frac{r^2}{4\nu t})]y, \quad \omega_2 = [-\frac{a_o}{\nu} + \frac{\gamma_o}{2}(\nu t)^{-\frac{3}{2}} \exp(-\frac{r^2}{4\nu t})]x, \quad (4.87)$$

which leads to a constant dynamic angle, namely

$$\phi = \arctan \frac{\beta_o}{\nu}. \quad (4.88)$$

The vorticity lies on the x, y -plane and is orthogonal to $\vec{r} = x\hat{x} + y\hat{y}$, i.e. $\vec{\omega} \cdot \vec{r} = 0$. The orientation of $\vec{\omega}$ with respect to S $\vec{\omega}$ depends on the ratio $\frac{\beta_o}{\nu}$; for $\beta_o = 0$, $\phi = 0$; for $\beta_o = \nu$, $\phi = \frac{\pi}{4}$, while in absence of viscosity ($\nu \rightarrow 0$) and when $\beta_o \neq 0$ we have $\phi = \frac{\pi}{2}$.

2) Let us consider the solution (4.68)-(4.69b). Owing to the complicated structure of u_3 (see (4.68)), the expression of the vorticity results too cumbersome. Therefore, it is convenient to examine the asymptotic behaviour of u_3 for $r \rightarrow \infty$ at fixed μ . By choosing $a_o = 0$, we get [11, p. 364]

$$u_3 \sim \exp(-\delta t) r^{\mu-\frac{1}{2}} [c_1 \cos(\sqrt{\frac{\delta}{\nu}} r - \mu \frac{\pi}{2} - \frac{\pi}{4}) + c_2 \sin(\sqrt{\frac{\delta}{\nu}} r - \mu \frac{\pi}{2} - \frac{\pi}{4})]. \quad (4.89)$$

Now the vorticity corresponding to the solution (4.69a), (4.69b) and (4.89) and the dynamic angle can be easily calculated. For simplicity, we report only the value of ϕ , which is

$$\phi = \arctan \frac{\beta_o}{\alpha_o}.$$

Thus, the discussion on this result is similar to that performed in the case 1). We observe that for $\alpha_o = \beta_o = 0$, ϕ is not defined.

An interesting case is represented by the choice $\mu = \frac{1}{2}$ in Eq. (4.89). In fact, we get

$$u_3 \sim \exp(-\frac{\Omega^2}{2}\tau^2)(c_1 \sin \Omega t - c_2 \cos \Omega t), \quad (4.90)$$

where $\Omega^2 = 2\delta$ and $\tau = t^{\frac{1}{2}}$. Formally, the quantity (4.90) can be interpreted as a special case of the general solution of the parametric oscillator described by the equation

$$\ddot{q} + 4\gamma_1 t \dot{q} + (\gamma_2^2 + 2\gamma_1 + 4\gamma_1^2 t^2)q = 0, \quad (4.91)$$

whose general solution is

$$q = \exp(-\gamma_1 t^2)(c_1 \cos \gamma_2 t + c_2 \sin \gamma_2 t), \quad (4.92)$$

where the amplitude plays the role of a damping function ($\gamma_1 > 0$) which tends to zero for large values of time according to a law of the Gaussian type.

3) The dynamic angles for the Burgers and Burgers-Lundgren vortices, together with the vortex solution with $f(r, t)$ given by (4.86), are zero.

5 Invariance of the boundary conditions

The study of the invariance properties of a system of differential equations with suitable boundary conditions plays a fundamental role in the description of realistic models.

Generally, boundary value problems can be treated more simply for ordinary differential equations. Indeed, a symmetry analysis allows one to reduce a given boundary value problem for an ordinary differential equation to a boundary value problem for a (reduced) equation of lower order. In the case of a partial differential equation, by a group point of view, a boundary value problem is solved when one can determine a solution which is invariant, together with all the boundary conditions, under the action of the infinitesimal generator of the symmetry group. However, for a linear partial differential equation one can resort to less restrictive conditions [8, p. 215].

In this Section we present a systematic investigation of all the surfaces which can take the meaning of invariant boundary surfaces in a boundary value problem for the Navier-Stokes (1.1)–(1.2).

To this aim, we recall that if V denotes a generator of symmetry transformations, a basis of invariants for V is obtained from the equation

$$VI = 0. \quad (5.1)$$

Now let $V = \sum_{i=1}^9 V_i$ be the vector field generating the set of the Lie–point transformations of Eqs. (1.1)–(1.2). Using the method of characteristics, Eq. (5.1) can be written in the form

$$\begin{aligned} \frac{dx}{ax + by + ez + g_o} &= \frac{dy}{ay - bx + cz + h_o} = \frac{dz}{az - cy - ex + r_o} = \\ \frac{dt}{2at + d} &= \frac{du_1}{-au_1 + bu_2 + eu_3} = \frac{du_2}{-au_2 - bu_1 + cu_3} = \\ \frac{du_3}{-au_3 - cu_2 - eu_1} &= \frac{dp}{k_o - 2ap}. \end{aligned} \quad (5.2)$$

Here we have put $g = g_o = \text{const}$, $h = h_o = \text{const}$, $k = k_o = \text{const}$. By solving the system (5.2), we can determine the characteristic manifolds $\lambda = \text{const}$, $\chi = \text{const}$, $\psi = \text{const}$ in the coordinate space (x, y, z, t) . To this aim,

first let us analyse the general case. Subsequently, some special interesting case will be examined.

5.1 The general case

The system (5.2) can be put into the form:

$$\frac{d\vec{x}}{d\tau} = A \cdot \vec{x} + \vec{r} \quad (5.3)$$

where:

$$A = \begin{pmatrix} a & b & e \\ -b & a & c \\ -e & -c & a \end{pmatrix} \quad (5.4)$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec{r} = \begin{pmatrix} g_0 \\ h_0 \\ r_0 \end{pmatrix} \quad (5.5)$$

$$\tau = \frac{1}{2a} \ln \left(\frac{2at}{d} + 1 \right). \quad (5.6)$$

Let us distinguish the following cases.

Case I) $a \neq 0$.

Eq. (5.3) admits the solution

$$\vec{x} = e^{A\tau} \vec{x}_0 + \vec{x}_P, \quad (5.7)$$

where \vec{x}_0 is a constant vector determined by initial conditions, and

$$\vec{x}_P = -A^{-1} \vec{r} \quad (5.8)$$

First, let us consider the subcase $\vec{r} = 0$.

The solution of the system (5.3) (in the coordinate space), is provided by the following invariants:

$$\begin{aligned}
\lambda = & \frac{1}{\sqrt{2a\frac{t}{d}+1}} \left\{ \left(\frac{1}{\omega^2} [c^2 + (b^2 + e^2) \cos \omega \tau] \right) x + \right. \\
& \left(\frac{ce}{\omega^2} [-1 + \cos \omega \tau] - \frac{b}{\omega} \sin \omega \tau \right) y + \\
& \left. \left(-\frac{bc}{\omega^2} [-1 + \cos \omega \tau] - \frac{e}{\omega} \sin \omega \tau \right) z \right\}, \quad (5.9)
\end{aligned}$$

$$\begin{aligned}
\mu = & \frac{1}{\sqrt{2a\frac{t}{d}+1}} \left\{ \left(\frac{ce}{\omega^2} [-1 + \cos \omega \tau] + \frac{b}{\omega} \sin \omega \tau \right) x + \left(\frac{1}{\omega^2} [c^2 + \right. \right. \\
& \left. \left. (b^2 + e^2) \cos \omega \tau \right] y + \left(\frac{be}{\omega^2} [-1 + \cos \omega \tau] - \frac{c}{\omega} \sin \omega \tau \right) z \right\}, \quad (5.10)
\end{aligned}$$

$$\begin{aligned}
\nu = & \frac{1}{\sqrt{2a\frac{t}{d}+1}} \left\{ \left(-\frac{bc}{\omega^2} [-1 + \cos \omega \tau] + \frac{e}{\omega} \sin \omega \tau \right) x + \right. \\
& \left(\frac{be}{\omega^2} [-1 + \cos \omega \tau] + \frac{c}{\omega} \sin \omega \tau \right) y + \\
& \left. \left(\frac{1}{\omega^2} [b^2 + (c^2 + e^2) \cos \omega \tau] \right) z \right\}, \quad (5.11)
\end{aligned}$$

where $\omega = \sqrt{b^2 + c^2 + e^2}$.

From general formulas (5.9)–(5.11) we can obtain the following invariants, in a simple form:

$$\lambda = \sqrt{\frac{d}{2at+d}} (cx - ey + bz) \quad (5.12)$$

$$\mu = \sqrt{\frac{d}{2at+d}} (ex + cy)^2 + b^2(x^2 + y^2) + (c^2 + e^2)z^2 + b(-2cxz + 2eyz) \quad (5.13)$$

$$\nu = \frac{(ex + cy)^2 + b^2(x^2 + y^2) + (c^2 + e^2)z^2 + b(-2cxz + 2eyz)}{(cx - ey + bz)^2} \quad (5.14)$$

The case where $g_o \neq 0, h_o \neq 0, r_o \neq 0$ can be obtained from the previous one by making the change of variables:

$$x \rightarrow x + \frac{a^2 g_o + c(cg_o - eh_o + br_o) - a(bh_o + er_o)}{a(a^2 + b^2 + c^2 + e^2)}, \quad (5.15)$$

$$y \rightarrow y + \frac{abg_o - ceg_o + (a^2 + e^2)h_o - (ac + be)r_o}{a(a^2 + b^2 + c^2 + e^2)}, \quad (5.16)$$

$$z \rightarrow z + \frac{bcg_o + aeg_o + (ac - be)h_o + (a^2 + b^2)r_o}{a(a^2 + b^2 + c^2 + e^2)}. \quad (5.17)$$

Case II) $a = 0$.

For $\vec{r} = 0$ we obtain the following invariants:

$$\lambda = (cx - ey + bz) \quad (5.18)$$

$$\mu = x^2 + y^2 + z^2 \quad (5.19)$$

$$\sqrt{b^2 + c^2 + e^2} \frac{t}{d} - \pi \arctan \frac{ex + cy}{b(-cx + ey) + (c^2 + e^2)z} \quad (5.20)$$

Notice that λ represents an invariant plane, whereas μ is an invariant spheric surface centered at the origin.

For $g_o \neq 0, h_o \neq 0, r_o \neq 0$, we obtain:

$$\lambda = cx - ey + bz - \frac{t}{d}(cg_o - eh_o + br_o) \quad (5.21)$$

$$\begin{aligned} \mu = & \frac{1}{\sqrt{c^2 + e^2}} \cos(\omega \frac{t}{d}) \left(-\frac{bc}{\omega^2} g_0 + \frac{be}{\omega^2} h_0 + \frac{c^2 + e^2}{\omega^2} r_0 - ex - cy \right) + \\ & \frac{1}{\sqrt{c^2 + e^2}} \sin \omega \frac{t}{d} \left(\frac{e}{\omega} g_0 + \frac{c}{\omega} h_0 - \frac{bc}{\omega} x + \frac{be}{\omega} y + \frac{c^2 + e^2}{\omega} z \right) \end{aligned} \quad (5.22)$$

$$\nu = \left(\frac{e}{\chi} x + \frac{c}{\chi} y - \frac{bcg_0 - beh_0 - \chi^2 r_0}{\omega^2 \chi} \right)^2 + \left(-\frac{bc}{\omega \chi} x + \frac{be}{\omega \chi} y + \frac{\chi}{\omega} z + \frac{eg_0 + ch_0}{\omega \chi} \right)^2. \quad (5.23)$$

Eq. (5.23) represents a cylinder with radius

$$r = |\lambda|$$

and axis parallel to vector

$$\frac{c}{\omega} \hat{x} - \frac{e}{\omega} \hat{y} + \frac{b}{\omega} \hat{z} \quad (5.24)$$

passing through the point of coordinates:

$$\begin{aligned} & \frac{2bceg_0 + b(c^2 - e^2)h_0 - e\chi^2r_0}{\omega^2\chi^2}, \frac{g_0b(c^2 - e^2) - 2bceh_0 - cr_0\chi^2}{\omega^2\chi^2}, \\ & -\frac{eg_0 + ch_0}{\omega^2} \end{aligned} \quad (5.25)$$

In the space of velocities, the invariant curves, which are now functions of u_1, u_2, u_3 and p , can be deduced by the previous ones by performing the substitution

$$a \rightarrow -a, x \rightarrow u_1, y \rightarrow u_2, z \rightarrow u_3, t \rightarrow p, d \rightarrow k_o. \quad (5.26)$$

At this point let us examine some particular cases which emerge by demanding that some arbitrary constants appearing in (5.2) are vanishing. Precisely, we shall distinguish the cases:

Subcase I: $\xi_4 \neq 0$, where

I1 : $d \neq 0, a = b = c = e = 0$,

I2 : $d \neq 0, a = b = e = 0, c \neq 0$,

I3 : $d \neq 0, a = b = 0, c \neq 0, e \neq 0$,

I4 : $d \neq 0, a = 0, b \neq 0, c \neq 0, e \neq 0$.

Subcase II: $\xi_4 = 0$, with $a = d = 0$, and

II1 : $b = e = c = 0$,

- II2 : $b \neq 0, e = c = 0,$
- II3 : $b \neq 0, c \neq 0, e = 0,$
- II4 : $b \neq 0, c \neq 0, e \neq 0.$

The expressions for the invariants λ, χ and ψ corresponding to the sub-cases II1,...,I4 and II1,...,II4 are given by

Subcase I1

$$\lambda = t - \frac{d}{g_o}x, \quad (5.27)$$

$$\chi = y - \frac{h_o}{g_o}x, \quad (5.28)$$

$$\psi = z - \frac{r_o}{g_o}x. \quad (5.29)$$

Subcase I2

$$\lambda = t - \frac{d}{g_o}x, \quad (5.30)$$

$$\chi = \frac{h_o + cz}{c} \cos \frac{cx}{g_o} + \frac{cy - r_o}{c} \sin \frac{cx}{g_o}, \quad (5.31)$$

$$\psi = \frac{cy - r_o}{c} \cos \frac{cx}{g_o} - \frac{h_o + cz}{c} \sin \frac{cx}{g_o}. \quad (5.32)$$

Subcase I3

$$\lambda = -\frac{c}{e}x + y, \quad (5.33)$$

$$\chi = x^2 + y^2 + z^2, \quad (5.34)$$

$$\psi = t + \frac{d}{\sqrt{c^2 + e^2}} \arctan \frac{z\sqrt{c^2 + e^2}}{ex + cy}. \quad (5.35)$$

Subcase I4

$$\lambda = \frac{cx - ey}{b} + z, \quad (5.36)$$

$$\chi = x^2 + y^2 + z^2, \quad (5.37)$$

$$\psi = t - \frac{d}{\sqrt{b^2 + c^2 + e^2}} \arctan \frac{b^2 x + e(ex + cy) - bc z}{\sqrt{b^2 + c^2 + e^2}(by + ez)}. \quad (5.38)$$

Subcase II1

$$\lambda = t, \quad (5.39)$$

$$\chi = y - \frac{h_o}{g_o} x, \quad (5.40)$$

$$\psi = z - \frac{r_o}{g_o} x. \quad (5.41)$$

Subcase II2

$$\lambda = t, \quad (5.42)$$

$$\chi = (x - \frac{h_o}{b}) \cos \frac{b}{r_o} z - (y + \frac{g_o}{b}) \sin \frac{b}{r_o} z, \quad (5.43)$$

$$\psi = (x - \frac{h_o}{b}) \sin \frac{b}{r_o} z + (y + \frac{g_o}{b}) \cos \frac{b}{r_o} z. \quad (5.44)$$

Subcase II3

$$\lambda = t, \quad (5.45)$$

$$\chi = z + \frac{c}{b} x, \quad (5.46)$$

$$\psi = x^2 + y^2 + z^2. \quad (5.47)$$

Subcase II4

$$\lambda = t, \quad (5.48)$$

$$\chi = cx - ey + bz, \quad (5.49)$$

$$\psi = x^2 + y^2 + z^2. \quad (5.50)$$

6 Conclusions

We summarize the main results achieved in this work. Using a group-theoretical framework, we have established a connection between the symmetry generators of the incompressible 3D Navier-Stokes equations and the existence of solutions having a vortical dynamics. Precisely, by means of a procedure of symmetry reduction of Eqs. (1.1)–(1.2), we have found some novel (at the best of our knowledge) exact solutions which contain as special cases important solutions of the vortex-type well-known in the literature, as the Burgers vortex and shear-layer solutions. Such solutions have been discussed by the point of view of their possible physical meaning.

It is suitable to recall that an approach of the algebraic and group-theoretical type to the equations for the fluid dynamics, especially to the incompressible Navier-Stokes equations, is not new and has been carried out by many authors (see, for example, [10], and [17]–[22]). In particular, the case of two space variables have been investigate more frequently and in more detailed manner. This is due to the fact that in 2D dimensions the possibility of defining a stream function [17] simplifies the mathematical treatment of the problem. In this spirit, recently Ludlow, Clarkson and Bassom [17] have obtained non-classical symmetries for the 2D Navier-Stokes equations applying the so-called direct method [23]. On the other hand, in [20] Fushchich, Shtelen and Slavutsky have determined solutions of the

3D Navier-Stokes equations performing a systematic study of the inequivalent ansaetze of codimension 1 by means of which a direct reduction of the equations under consideration to ordinary differential equations is obtained.

In our paper, together with the finding of exact solutions linked to the reduction method and together with the analysis of the role that the symmetry infinitesimal operators plays in understanding the vortical phenomena, we have classified in the context of the group theory all the invariant surfaces related to boundary conditions of physical interest. This kind of research is of special utility in possible applications to problems where the temperature is present. In particular, we mention the case in which to the Eqs. (1.1)–(1.2) one adds a transport equation describing the behavior of the temperature. These equations are known as the Navier–Stokes–Fourier equations, where the real coupling among them occurs just via the boundary conditions [24].

To conclude, we notice that in the symmetry algebra commutators associated with the 3D Navier-Stokes Eqs. (1.1)–(1.2) arbitrary functions of integration appear. This indicates that the algebra is infinite-dimensional. Hence, it should be interesting to explore deeply this algebra and to classify, for example, its one-dimensional subalgebras in order to discover other possible solutions of the 3D Navier–Stokes equations.

Appendix A: the symmetry algebra

The appearance of arbitrary functions in the vector fields (2.7)–(2.15) ensures that the symmetry algebra of the Navier-Stokes equations is infinite-dimensional. This algebra is defined by the commutation relations

$$[V_1(g_1), V_1(g_2)] = V_4(g_2 \ddot{g}_1 - g_1 \ddot{g}_2), \quad [V_2(h_1), V_2(h_2)] = V_4(h_2 \ddot{h}_1 - h_1 \ddot{h}_2), \quad (A.1)$$

$$[V_3(r_1), V_3(r_2)] = V_4(r_2 \ddot{r}_1 - r_1 \ddot{r}_2), \quad [V_4(k_1), V_4(k_2)] = 0, \quad (A.2)$$

$$[V_1(g), V_2(h)] = 0, \quad [V_1(g), V_3(r)] = 0, \quad (A.3)$$

$$[V_1(g), V_4(k)] = 0, \quad [V_1(g), V_5(a)] = V_1(ag - 2a \dot{g} t), \quad (A.4)$$

$$[V_1(g), V_6(b)] = -V_2(bg), \quad [V_1(g), V_7(c)] = 0, \quad (\text{A.5})$$

$$[V_1(g), V_8(d)] = -V_3(dg), \quad [V_1(g), V_9(e)] = -V_1(e \dot{g}), \quad (\text{A.6})$$

$$[V_2(h), V_3(r)] = 0, \quad [V_2(h), V_4(k)] = 0, \quad (\text{A.7})$$

$$[V_2(h), V_5(a)] = V_2(ah - 2a \dot{h} t), \quad [V_2(h), V_6(b)] = V_1(bh), \quad (\text{A.8})$$

$$[V_2(h), V_7(c)] = -V_3(ch), \quad [V_2(h), V_8(d)] = 0, \quad (\text{A.9})$$

$$[V_2(h), V_9(e)] = -V_2(e \dot{h}), \quad [V_3(r), V_4(k)] = 0, \quad (\text{A.10})$$

$$[V_3(r), V_5(a)] = V_3(ar - 2a \dot{r} t), \quad [V_3(r), V_6(b)] = 0, \quad (\text{A.11})$$

$$[V_3(r), V_7(c)] = V_2(cr), \quad [V_3(r), V_8(d)] = V_1(dr), \quad (\text{A.12})$$

$$[V_3(r), V_9(e)] = -V_3(e \dot{r}), \quad [V_4(k), V_5(a)] = -2V_4(ak + a \dot{k} t), \quad (\text{A.13})$$

$$[V_4(k), V_6(b)] = 0, \quad [V_4(k), V_7(c)] = 0, \quad (\text{A.14})$$

$$[V_4(k), V_8(d)] = 0, \quad [V_4(k), V_9(e)] = -V_4(e \dot{k}), \quad (\text{A.15})$$

$$[V_5(a), V_6(b)] = 0, \quad [V_5(a), V_7(c)] = 0, \quad (\text{A.16})$$

$$[V_5(a), V_8(d)] = 0, \quad [V_5(a), V_9(e)] = -2V_9(ae), \quad (\text{A.17})$$

$$[V_6(b), V_7(c)] = -V_8(bc), \quad [V_6(b), V_8(d)] = V_7(bd), \quad (\text{A.18})$$

$$[V_6(b), V_9(e)] = 0, \quad [V_7(c), V_8(d)] = -V_6(cd), \quad (\text{A.19})$$

$$[V_7(c), V_9(e)] = 0, \quad [V_8(d), V_9(e)] = 0. \quad (\text{A.20})$$

Scrutinizing these commutation rules, it emerges that they contain the subalgebras

$$\begin{aligned} & \{V_1, V_2, V_5\}, \{V_1, V_2, V_6\}, \{V_1, V_2, V_9\}, \{V_1, V_3, V_5\}, \{V_1, V_3, V_8\}, \{V_1, V_3, V_9\}, \\ & \{V_1, V_4, V_5\}, \{V_1, V_5, V_9\}, \{V_2, V_3, V_5\}, \{V_2, V_3, V_7\}, \{V_2, V_3, V_9\}, \{V_2, V_4, V_5\}, \\ & \{V_2, V_4, V_9\}, \{V_2, V_5, V_9\}, \{V_3, V_4, V_5\}, \{V_3, V_4, V_9\}, \{V_3, V_5, V_9\}, \{V_4, V_5, V_9\}, \\ & \{V_6, V_7, V_8\}, \end{aligned} \quad (\text{A.21})$$

which turn out to be isomorphic to the algebra of the Euclidean group E_2 .

Appendix B: derivation of the solution (4.41a)–(4.41d)

In order to find a particular solution of Eqs. (4.41a)–(4.41d), let us assume that

$$\Lambda_1 = h(\theta), \Lambda_2 = f(\theta), \Lambda_3 = g(\theta). \quad (\text{B.1})$$

Then, Eq. (4.41d) provides

$$\Lambda_1 = c\theta^{\frac{1}{2}}, \quad (\text{B.2})$$

c being a constant of integration, while Eqs. (4.41a)–(4.41c) give

$$2\pi - 2\theta\pi_\theta - \eta\pi_\eta - \xi\pi_\xi + \frac{1}{2}c\theta^{\frac{3}{2}} = 0, \quad (\text{B.3})$$

$$\pi_\xi = \frac{1}{\xi^2}F(\theta), \quad (\text{B.4})$$

$$\pi_\eta = \frac{1}{\eta^2}G(\theta), \quad (\text{B.5})$$

respectively, where

$$F(\theta) = -(2\nu + c\theta^{\frac{1}{2}})f + (2\nu\theta + 2c\theta^{\frac{3}{2}} - \theta^2)f' - 4\nu\theta^2f'', \quad (\text{B.6})$$

$$G(\theta) = -(2\nu + c\theta^{\frac{1}{2}})g + (2\nu\theta + 2c\theta^{\frac{3}{2}} - \theta^2)g' - 4\nu\theta^2g'', \quad (\text{B.7})$$

with $f' = \frac{df}{d\theta}$, $g' = \frac{dg}{d\theta}$.

By integrating Eq. (B.4) with respect to ξ , we get

$$\pi = -\frac{1}{\xi}F(\theta) + \alpha(\eta, \theta), \quad (\text{B.8})$$

where $\alpha(\eta, \theta)$ is a function of integration.

From (B.8) and (B.5) we have

$$\pi_\eta = \alpha_\eta = \frac{G(\theta)}{\eta^2}, \quad (\text{B.9})$$

from which

$$\alpha = -\frac{G(\theta)}{\eta} + \beta(\theta), \quad (\text{B.10})$$

where β is an arbitrary function of θ . Taking account of (B.10), Eq. (B.8) becomes

$$\pi = -\frac{F(\theta)}{\xi} - \frac{G(\theta)}{\eta} + \beta(\theta). \quad (\text{B.11})$$

Then, substitution from (B.11) into Eq. (B.3) yields

$$F = a\theta^{\frac{3}{2}}, \quad (\text{B.12})$$

$$G = b\theta^{\frac{3}{2}}, \quad (\text{B.13})$$

$$\beta = \frac{c}{2}\theta^{\frac{3}{2}} + c_0\theta, \quad (\text{B.14})$$

where a, b, c_0 are constants.

Now it is convenient to put $\theta^{\frac{1}{2}} = \rho$. So, with the help of (B.12), Eq. (B.6) can be written as

$$\nu\rho^2 f_{\rho\rho} + \left(\frac{1}{2}\rho^3 - c\rho^2 - 2\nu\rho\right)f_\rho + (c\rho + 2\nu)f + a\rho^3 = 0. \quad (\text{B.15})$$

The general solution of this equation is

$$f(\rho) = -2a\rho + \sqrt{\nu} \exp \frac{\rho(4c - \rho)}{4\nu} [c_1 + c_2 \operatorname{erf} \sqrt{-\frac{(2c - \rho)^2}{4\nu}}]. \quad (\text{B.16})$$

The form of $g(\rho)$ can be determined in the same way. Coming back to the original variable θ and exploiting (B.1), (B.2), (B.11), the solution (4.41a)–(4.41d) is readily found.

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